Model Order Reduction of Large Circuits Using Balanced Truncation Via the Arnoldi Method

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Outline

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Introduction

- In high-frequency range, circuits should be modeled as distributed elements
- Extracted circuits are huge and cannot be simulated without order reduction
- AWE is a method for order reduction based on Pade approximation of the system transfer function
- Improvements of AWE include RICE, PVL and PRIMA (guarantees passivity)

Overview

- Model order reduction has extensively been studied in the control engineering
- Both frequency- and time-domain model reduction techniques have been proposed
- An effective method is based on the balanced realization of the system
  - Guaranteed stability
  - Bound on the error for the reduced system
  - Provably optimal solution
State Space Form

- Any linear, time-invariant circuit can be written in standard state space form as:
  \[ \dot{x} = A\dot{x} + Bu \]
  \[ y = Cx + Du \]
- Given the state space matrix \((A, B, C, D)\), the transfer function of the system is:
  \[ G(s) = C(sI - A)^{-1}B + D \]
- Without loss of generality, assume \(D=0\)

Singular Value Decomposition

- Any \(l \times m\) matrix \(A\) may be factorized into a singular value decomposition \(A = U\Sigma V^T\) where the \(l \times l\) matrix \(U\) and the \(m \times m\) matrix \(V\) are unitary and the \(l \times m\) matrix \(\Sigma\) contains a diagonal matrix \(\Sigma\) of real, non-negative singular values \(\sigma\), arranged as \([\Sigma, 0]\) if \(l < m\), as \(\Sigma\), if \(l = m\) and as \([\Sigma, 0]^T\) otherwise.
  - Note that \(\Sigma = \text{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_k\}; k = \min(l, m)\)
  - and \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k\) the singular values \(\sigma_i\) are positive square roots of the \(k\) largest eigenvalues of both \(AA^T\) and \(A^TA\). Matrices \(U\) and \(V\) are unit eigen-vectors of \(AA^T\) and \(A^TA\).
Balancing the System

- The idea is to find a state vector realization of the system that results in equal coupling of energy from the inputs to states and from states to the outputs.

- The Reachability and Observibility Gramians are measures of such energy couplings:
  \[ W_r = \int_0^\infty e^{A\tau}BB^T e^{A^T\tau} d\tau \quad W_o = \int_0^\infty e^{A\tau}C^T Ce^{A^T\tau} d\tau \]

- The Gramians are obtained by solving the Lyapunov equations.

Lyapunov Equations

- The Lyapunov equations can be stated as follows:
  \[ AW_r + W_r A^T + BB^T = 0 \]
  \[ A^T W_o + W_o A + C^T C = 0 \]

- The Hankel singular values and the Hankel norm are then calculated as:
  \[ \sigma_i(G(s)) = \sqrt{\lambda_i(W_r W_o)}, i = 1, 2, \ldots, n \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \]
  \[ \|G(s)\|_H = \max \sigma_i = \sigma_1 \]
Time-Domain View of Hankel Norm

- It can be shown that Hankel norm is also given by:

\[
\|G(s)\|_H = \max_{\hat{w}(t)} \left( \frac{\sqrt{\int_0^\infty \|\hat{y}(t)\|^2 dt}}{\sqrt{\int_{-\infty}^0 \|\hat{u}(t)\|^2 dt}} \right)
\]

- Hankel norm can be interpreted as a kind of induced norm from past inputs to future outputs

Balanced Realization (BR)

- By applying a transformation \(T\) to the system, we can change \(W_r\) and \(W_o\) as follows:

\[
\hat{W}_r = T^{-1} W_r T^{-T} \quad \hat{W}_o = T^T W_o T
\]

- It can be shown that for any system, there is a transformation which makes

\[
\hat{W}_r = \hat{W}_o
\]

- Using such a transformation, the new system is called a balanced realization
Reduced System

- To reduce order of the system, we simply ignore states with small Hankel singular values:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \ C_2], \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}
\]

- **Original System**

\[
G(s) = C(sI - A)^{-1}B
\]

- **Reduced System**

\[
G^k(s) = C_1(sI - A_{11})^{-1}B_1
\]

$H_\infty$ System Norm

- $H_\infty$ norm of the system is defined as:

\[
\|G(s)\|_\infty = \max_\omega \sigma_i(G(j\omega))
\]

- Note that given any matrix $A$, $\sigma_i$ is the defined as:

\[
\sigma_i(A) = \sqrt{\lambda_i(A^HA)} = \sqrt{\lambda_i(AA^H)}
\]

- $\sigma_1$ is the largest $\sigma_i$

- It can be shown that:

\[
\|G(s)\|_\infty = \max_{\ddot{u}(t) \neq 0} \frac{\|\dddot{y}(t)\|_2}{\|\ddot{u}(t)\|_2}
\]

- So when $\|G_1(s) - G_2(s)\|_\infty \approx 0$, the two systems are almost identical
Main Theorems

- Let $G(s)$ denote a stable rational transfer function of degree $n$ with Hankel singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let $G_k^k(s)$ denote the order $k$ reduction of this transfer function as defined previously. We have:

$$\left\| G(s) - G_k^k(s) \right\| \leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$$

- The minimum error for approximating $G(s)$ with an arbitrary transfer function $H(s)$ of degree $r < n$ is given by:

$$\left\| G(s) - H(s) \right\|_\infty \geq \sigma_{k+1}$$

Numerical Methods

- We can directly obtain the reduced order system without calculating the BR

- Procedure (Safanov’89):
  - Compute matrices $V_{L,k}$ and $V_{R,k}$ whose columns form bases for the right and left eigen-spaces of $W_r W_o$ associated with the big eigen-values $\sigma_1^r, \ldots, \sigma_i^r$
  - Set $E = V_{L,k}^T V_{R,k}$
  - Compute singular value decomposition $U_k \Sigma_k V_k^T = E$
  - Set $S_L = V_{L,k} U_k \Sigma_k^{1/2} \in \mathbb{R}^{n \times k}$
  - $S_R = V_{R,k} U_k \Sigma_k^{-1/2} \in \mathbb{R}^{n \times k}$
  - The reduced order system is given by:

$$\hat{A} = S_L^T A S_L \quad \hat{B} = S_L^T B \quad \hat{C} = C S_L \quad \hat{D} = D$$
Numerical Methods

- In Safanov’s algorithm, we need $W_r$ and $W_o$ and then the Schur decomposition of $W_rW_o$ to obtain $V_{L,k}$ and $V_{R,k}$

- Large Lyapunov equations can be solved directly using Krylov-subspace methods (based on the Arnoldi algorithm) as shown in [Saad’89]

- For the decomposition of $W_rW_o$, we resort again to the Arnoldi algorithm

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Balanced Truncation Via Arnoldi

- Procedure BTVA
  - Use Krylov-subspace method to calculate $W_rW_o$
  - Use Arnoldi algorithm to calculate big eigenvalues and corresponding left and right eigenvectors $(V_{L,k}, V_{R,k})$ of $W_rW_o$
  - Choose the degree for reduced order system based on calculated eigenvalues and the desired error bounds
  - Compute
    \[
    E = V_{L,k}^\top V_{R,k} \quad U_r \Sigma_k V_k^\top = E
    \]
    \[
    S_L = V_{L,K}U_r \Sigma_k^{1/2} \in \mathbb{R}^{n \times k} \quad S_R = V_{R,K} \Sigma_k^{1/2} \in \mathbb{R}^{n \times k}
    \]
  - Compute the reduced order system
    \[
    \hat{A} = S_L^\top AS_R \quad \hat{B} = S_L^\top B \quad \hat{C} = CS_R \quad \hat{D} = D
    \]
Experimental Results

- Small Circuit:
  - 8->4 identical
  - 8->2 Some error

![Circuit Diagram]

Experimental Result

- Big System
  - 320-> 4 : Identical

![Circuit Diagram]
Conclusions

- Balanced realization is a provably optimal solution to order reduction of LTI systems
- It results in better reduced system compared to the Pade-based techniques
- The computational complexity may however limit the application of this method
- Better methods for solving Lyapunov equations are required to handle higher order systems